Gauss's Law: Teaching Platform Using the ***Magic Cube*** Implementation by COMSOL Multiphysics

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Abstract: This paper presents a *"Gauss's law"* teaching platform based on the use of a *"magic cube"*. The "*magic cube"* has been constructed from 5x5x5=125 small cubes; each having dimensions 1x1x1mm (i.e., size of $1mm^3$). This *"magic cube"* allows the construction of any arbitrary closed geometry simply by selecting appropriate surfaces using some of the internal cubes' boundaries. Similarly, arbitrary charge distributions in the form of volume, surface or linear charge densities, or even localized charges at some points are also easily defined inside the constructed *"magic cube"*. This allows students to deal with Gauss's law in practical cases where the spatial behavior of the electric flux density is not predictable. Students are also free to locate the charge inside or outside the selected closed surface and visualize the behavior and distribution of the electric flux density over the closed surface. Verification of Gauss's law becomes possible for any specified case, simply by integrating the electric flux density over the appropriate selected closed surface boundaries.

Keywords: Gauss's law, Gaussian surface, electric flux density, charge enclosed.

1. Introduction

Most probably Gauss's law is considered as the very first "*electromagnetic*" concept for early undergraduate physics and electromagnetic courses. In early study year, teaching Gauss's law is usually performed based on the use of some simple symmetrical charge distributions, where a correct expectation of the spatial behavior of the electric flux density is possible. Consequently, the definition of the "*Gaussian surface*", which is directly related to the charge distribution, becomes also a symmetrical topology. Finally, the surface integration of the electric flux density over the closed surface is evaluated using simple analytical formula. However, using such teaching methodology, students will be limited in the verification and understanding of Gauss's law only for simple symmetrical charge distributions and corresponding topologies.

At this stage, it would be beneficial to have a flexible structure that allows the verification of Gauss's law over any closed surface topology using arbitrary charge distribution, where the spatial behavior of the electric flux density is fully unpredictable. Using such reconfigurable platform, students will have unlimited degrees of freedom to define the type of charge distributions, its configuration, location, and finally construct a closed surface over which the integration of the electric flux density will be performed. Obviously, Gauss's law verification in this case will be performed without the need for a "*Gaussian surface*", which may not be available especially for non-symmetrical charge distributions topologies.

2. Gauss's law

The basic formulation for Gauss's law may be expressed by:

$$\oint_S D.dS = Q_{enclosed}$$

Where the integration is performed over a closed surface, D is the electric flux density and "$Q_{enclosed}$" represents the total charge enclosed within the selected closed surface.

Teaching Gauss's law starts by the definition of the charge distribution. Consequently, a correct expectation of the spatial behavior of the electric flux density will be required together with an appropriate surface over which the integration may be performed. The following two assumptions are usually implemented; 1) symmetrical charge distribution, where the spatial behavior of the electric flux density is well known, and 2) availability of a symmetrical surface called *"Gaussian surface"* where the electric flux density is always constant and perpendicular at this surface. Finally, the surface integration becomes simply an evaluation of the total surface area of the *"Gaussian surface"* multiplied by the constant value of the electric flux density.

Although seems a simple and direct technique, the above mentioned procedures becomes impractical and even impossible when dealing with arbitrary charge distributions. This is due to the fact that in such cases, the spatial behavior of the electric flux density is unpredictable, and the so called *"Gaussian surface"* may not be available at all.

3. The "Magic Cube"

The proposed teaching platform is based on the use of a *"magic cube"* that has been constructed using COMSOL. A *"magic cube"* of 5x5x5=125 small cubes, each of $1mm^3$ size is shown in Fig. 1.

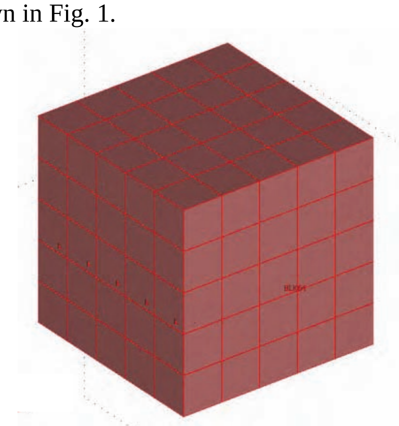


Figure 1: The constructed *"magic cube"* (125 cells, each of $1mm^3$ size)

Although the used *"magic cube"* consists of 125 small cubes, it is always possible to use a larger *"magic cube"* to more precisely model arbitrary closed surface. Moreover, the cell size, which is currently $1mm^3$, could be reduced to improve the accuracy of the results. However, these will be on the expenses of the required simulation resources and corresponding CPU time.

The flexibility of the *"magic cube"* as a teaching platform lies in the, almost; unlimited possibilities students have to construct any arbitrary charge distribution, without a prior knowledge of the spatial behavior of the electric flux density. Students can also define any arbitrary closed surface over which the integration of the electric flux density is to be performed. The huge scenarios offered by the proposed *"magic cube"* make it possible for students to handle Gauss's law in real practical situations and visualize field quantities over selected boundaries. This in turns help students understands the behavior of the electric flux lines due to non-symmetrical charge distributions.

Examples of non-symmetrical volume, and surface charge distributions constructed using the proposed *"magic cube"* is shown in Fig. 2.

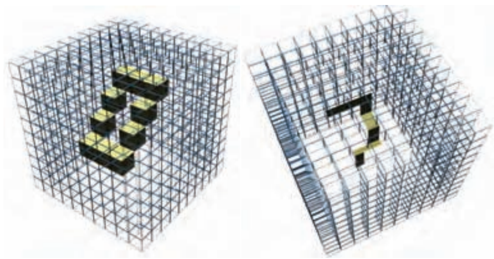


Figure 2: Examples of non-symmetrical charge distribution topologies (Magic cube of 1000 cells)

Similarly, some examples of arbitrary closed surfaces which are built using the proposed reconfigurable "*magic cube*" is shown in Fig. 3.

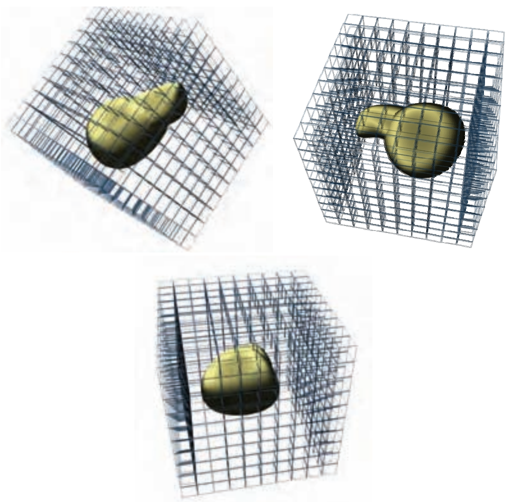


Figure 3: Examples of arbitrary constructed closed geometries (Magic cube of 1000 cells)

4. Implementation using COMSOL

The proposed platform; *"magic cube"* has been used to verify Gauss's law for some non-symmetrical charge distribution. A surface charge distribution of $1x10^{-6}$ C/m^2 has been used over a U-shaped surface as shown in Fig. 4, resulting in a total charge of $3x10^{-12}$C. This surface has been constructed from three internally connected surface boundaries, and is positioned in the center of the *"magic cube"* as shown in Fig. 5.

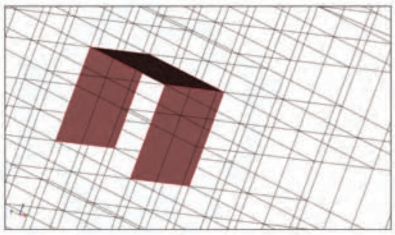


Figure 4: U-shaped surface
(Total charge $3x10^{-12}$C)

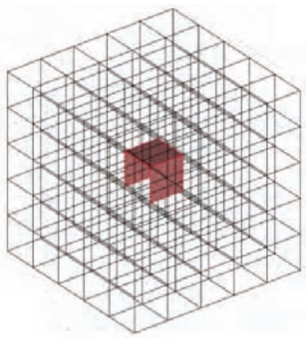


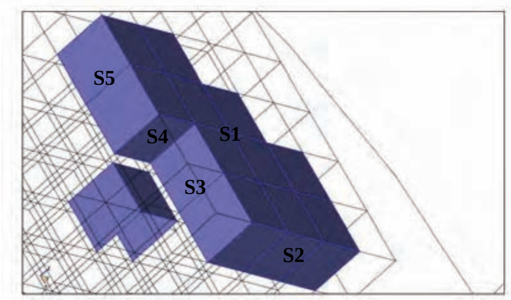
Figure 5: U-shaped surface positioned in the center of the *"magic cube"*

The whole structure of the *"magic cube"* has been constructed and simulated using the AC/DC 3D electrostatic mode. In order to properly build and simulate the model, the 5x5x5mm^3 *"magic cube"* has been enclosed in the center of an air sphere of radius 10cm.

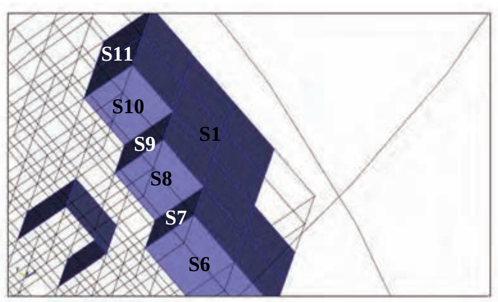
All boundaries of the *"magic cube"* have been set as 'continuity', while the outer sphere has been set as 'ground'. The U-shaped surface having the charge density has been set as 'charge density'. The selected set of boundary conditions ensures that any constructed closed surface could be used in the verification of Gauss's law.

4.1 Case I: Charge enclosed = zero

In this case, the closed surface has been constructed such that the U-shaped surface charge is excluded. This mean that the net charge enclosed "$Q_{enclosed}$" is zero. The selected surface is shown in Fig. 6 (a) & (b), where different faces are presented.



(a)



(b)

Figure 6: Closed surface excluding the charge

The electric flux density at some boundary surfaces of the closed structure is presented in Fig. 7, together with the total flux crossing the corresponding surface.

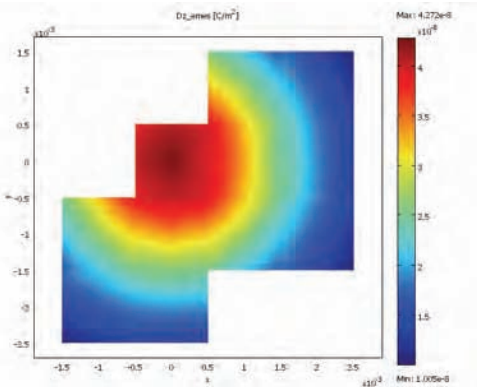


Figure 7 (a): Electric flux density at: S1 "Upper side"
(Total flux: 2.756712e-13 C)

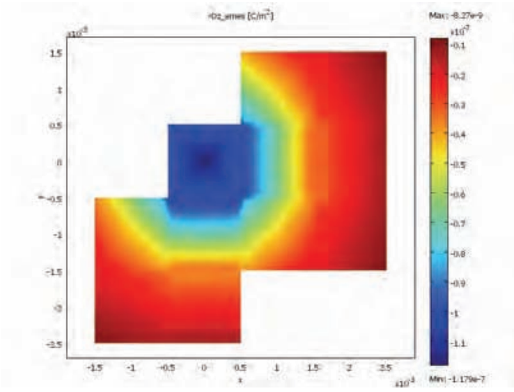


Figure 7 (b): Electric flux density at: S1 “Lower side”
(Total flux: -4.80971e-13 C)

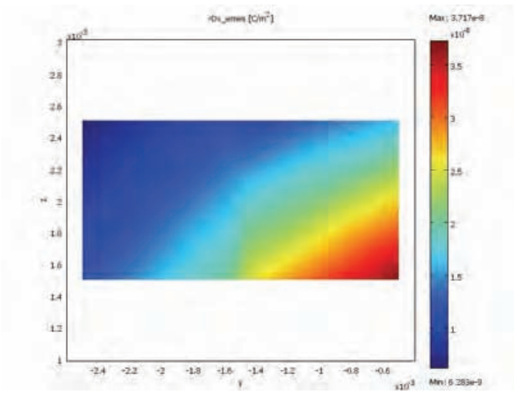
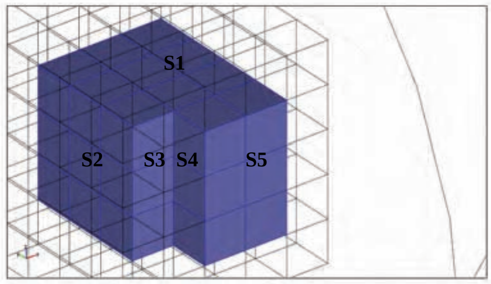


Figure 7 (c): Electric flux density at: S6
(Total flux: 3.448239e-14 C)

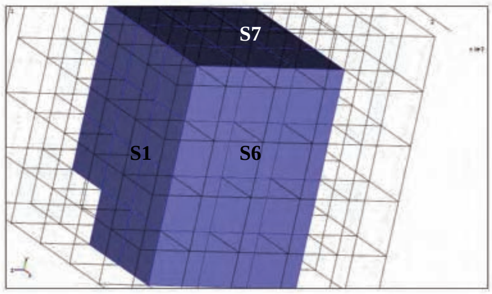
Once the behavior and distribution of the electric flux density is obtained, direct integration over the corresponding boundary surface gives the total flux crossing such boundary. Care should be taken in the integration process to account for the phase difference between the surface area vector and the electric flux density vector. The value of the total charge enclosed is finally determined by simply adding the total flux crossing all surfaces that form the closed structure. Students can easily verify that in the studied case, the total flux adds to give zero net charge enclosed.

4.2 Case II: Charge enclosed ≠ zero

In this case, the closed surface has been built with the U-shaped surface charge fully included, where the total charge enclosed is $3x10^{-12}$C. The constructed surface is shown in Fig. 8 (a) & (b), where different faces are presented.



(a)



(b)

Figure 8: Closed surface including the charge

Similar to the previous case, the electric flux density at some boundary surfaces of the closed structure is presented in Fig. 9, together with the total flux crossing the corresponding surface.

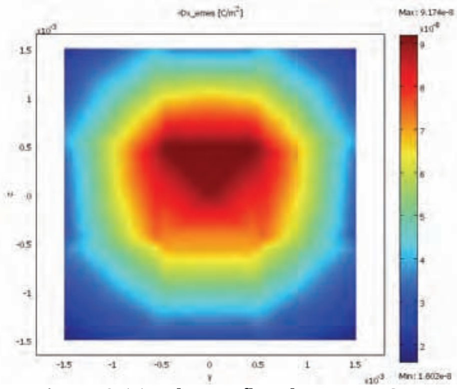


Figure 9 (a): Electric flux density at: S2
(Total flux: 4.842241e-13C)

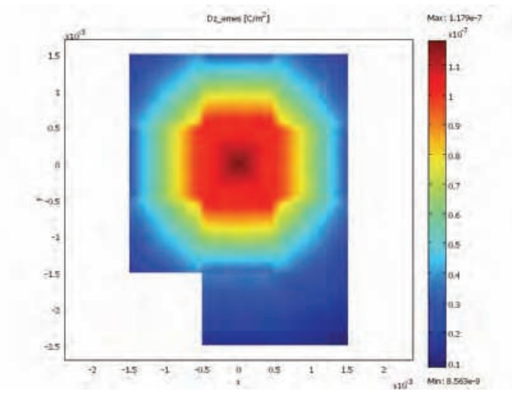


Figure 9 (b): Electric flux density at: S1 (Total flux: 6.032859e-13C)

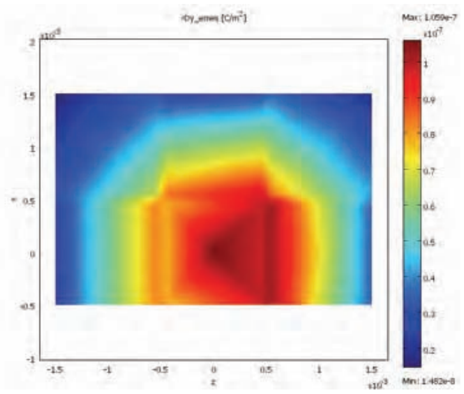


Figure 9 (c): Electric flux density at: S5 (Total flux: 3.648737e-13C)

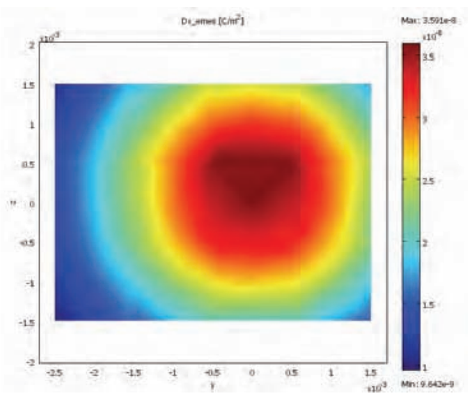


Figure 9 (d): Electric flux density at: S6 (Total flux: 2.984649e-13C)

Students can continue to determine the behavior of the electric flux density over the rest of boundaries forming the closed structure shown in Fig. 8, and calculate the total flux crossing each surface. Adding up the flux at all boundaries will result in the total charge enclosed within the surface. Students can perform the above exercise on any other closed surface, and verify Gauss's law in each case.

5. Conclusions

Using a reconfigurable "*magic cube*", which has been constructed using COMSOL, a flexible platform for teaching and understanding Gauss's law has been developed. It allows students to visualize the spatial distribution of the electric flux density due to arbitrary charge distribution topologies over any specified closed geometry. Evaluation of the total flux crossing the closed surface boundaries is achieved without the need for a "*Gaussian surface*", which may be an impossible task in practical situations. Consequently, students can now verify and assimilate Gauss's law for almost unlimited topologies, without the constraints imposed by symmetrical topologies, previously used in standard teaching methods.

6. References

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